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B-Decay CP Asymmetries, Discrete Ambiguities and New Physics

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Abstract: The first measurements of CP violation in the B system will likely probe $\sin 2\alpha$, $\sin 2\beta$ and $\cos 2\gamma$. Assuming that the CP angles α, β and γ are the interior angles of the unitarity triangle, we show that these measurements determine the angle set (α, β, γ) except for a twofold discrete ambiguity. If one allows for the possibility of new physics, the presence of this discrete ambiguity can make its discovery difficult: if only one of the two candidate solutions is consistent with constraints from other measurements in the B and K systems, one is not sure whether new physics is present or not. We examine the values of (α, β, γ) and the new-physics parameters for which this scenario can arise, and discuss ways to resolve the discrete ambiguity. The discrete ambiguity resolution often, but not always, helps to uncover the new physics.

1 Introduction

In the Standard Model (SM), CP violation is due to nonzero complex phases in the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix V. These CKM phases are elegantly described in terms of the interior angles α, β and γ of the "unitarity triangle" [1]. This triangle has two possible orientations. If we take the possible range of any angle to be $-\pi$ to $+\pi$, then, for one of these orientations, α, β and γ are all positive, while for the other they are all negative. Either way,

$$\alpha,\ \beta \text{ and } \gamma \text{ are all of the same sign.} \tag{1}$$

In addition, the angles in the unitarity triangle obviously satisfy the constraint

$$|\alpha+\beta+\gamma| = \pi\ . \tag{2}$$

The angles α, β and γ may be expressed in terms of the parameters ρ and η in Wolfenstein's approximation to the CKM matrix [2]. Existing information on $|V_{cb}|$, $|V_{ub}/V_{cb}|$, B_d and B_s mixing, and CP violation in the kaon system (ϵ_K) restricts ρ and η to the 95% confidence level allowed region shown in Fig. 1 [3]. Correspondingly, α, β and γ are restricted to the ranges

$$65^\circ \leq \quad \alpha \quad \leq 123^\circ\ , \tag{3}$$

$$16^\circ \leq \quad \beta \quad \leq 35^\circ\ , \tag{4}$$

$$37^\circ \leq \quad \gamma \quad \leq 97^\circ\ . \tag{5}$$

Note that the unitarity triangle shown in Fig. 1 points up, which implies that the CP angles α, β and γ are all positive. This is a consequence of the measured phase of ϵ_K and of the assumption that the kaon bag parameter $B_{\scriptscriptstyle K}$ is positive [4, 5]. While lattice calculations firmly indicate that, in fact, $B_{\scriptscriptstyle K} > 0$, this has not been verified experimentally. If $B_{\scriptscriptstyle K} < 0$, the unitarity triangle shown in Fig. 1 points down, so that the CP angles are all negative, with the above allowed ranges changing sign as well. In this paper we assume, as usual, that the lattice prediction that $B_{\scriptscriptstyle K} > 0$ is correct, but we shall comment on consequences for the search for new physics if it is not.

To test the SM picture of CP violation, and to look for evidence of new physics beyond the SM, coming experiments will attempt to determine the angles α, β and γ implied by CP-violating asymmetries in various B decays [6]. If the measured angles violate either of the "triangle conditions," Eqs. (1) and (2), or correspond to a point (ρ,η) which is outside the allowed region (Fig. 1), then we will have evidence of new physics.

The sign of the CP asymmetries in B_d decays is dependent on the sign of the bag parameter B_{B_d} [5]. Like $B_{\scriptscriptstyle K}$, B_{B_d} is firmly predicted by lattice calculations to be positive, and

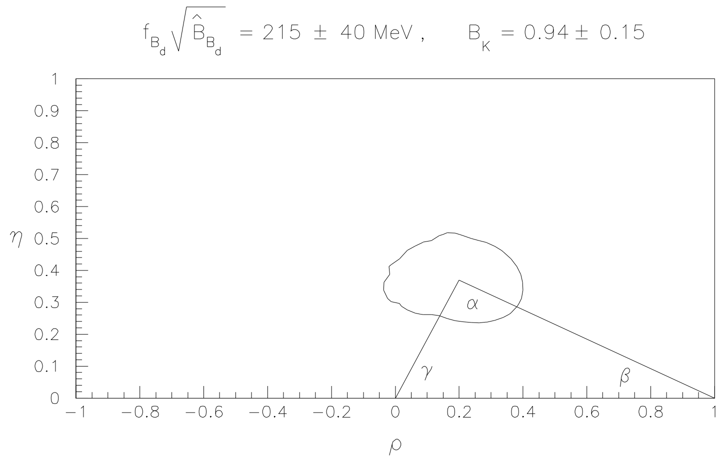

Figure 1: Allowed region (95% C.L.) in ρ–η space in the SM [3].

is usually assumed to have this sign. Under this assumption, if measurements yield a unitarity triangle which points downward, i.e. one which is inconsistent with the measurement of ϵ_K, this implies that new physics is present, even if the angles of the downward-pointing triangle have magnitudes consistent with Eqs. (3)-(5). The CDF collaboration has recently reported that, at the 93% confidence level, $\Gamma[\overline{B_d}(t) \to \Psi K_S] > \Gamma[B_d(t) \to \Psi K_S]$ [7]. If we combine this result with the constraint of Eq. (4) on $|\beta|$, and assume that $B_{B_d} > 0$, this result implies that the unitarity triangle points upward, consistent with the implication of ϵ_K.

Unfortunately, the CP asymmetries in the B system do not directly determine the angles in the unitarity triangle. Rather, these asymmetries yield only trigonometric functions of these angles, such as $\sin 2\alpha$ and $\sin 2\beta$, leaving the underlying angles themselves discretely ambiguous [8]. Needless to say, this makes the goal of testing for consistency with the SM and looking for evidence of physics beyond it more challenging. The purpose of this paper is to identify the discrete ambiguities which will be encountered even when reasonable constraints are imposed, to show what it will take to resolve these discrete ambiguities, and to see how their resolution will help (and sometimes not help) in testing the SM and in revealing physics beyond it.

The first CP asymmetry to be measured will almost certainly be the one in $B_d^0(t) \to \Psi K_S$. The second may well be the one in $B_d^0(t) \to \pi^+\pi^-$ (or $\pi^+\pi^-\pi^0$ [9]). Quite possibly, the third will be the one in $B^\pm \to DK^\pm$ [10]. We shall assume this scenario. Now, the CP asymmetry in any B decay probes the CP-odd part of the relative phase of (hopefully only) two interfering amplitudes. We shall call the CP-odd relative phases probed in $B_d^0(t) \to \pi^+\pi^-$, $B_d^0(t) \to \Psi K_S$ and $B^\pm \to DK^\pm$, respectively, $2\tilde{\alpha}$, $2\tilde{\beta}$ and $\tilde{\gamma}$.[1] The

[1]In $B_d^0(t) \to \pi^+\pi^-$ there may be significant penguin contributions [11], so that there are more than two interfering amplitudes. If this is the case, we assume that an isospin analysis is employed to find the

Process	Relative Phase of Amplitudes	CP Asymmetry Measures	Value of Probed Phase
$B_d^0(t) \to \pi^+\pi^-$	$2\tilde{\alpha}$	$\sin 2\tilde{\alpha}$	$\tilde{\alpha} = \alpha + \theta_d$
$B_d^0(t) \to \Psi K_S$	$2\tilde{\beta}$	$\sin 2\tilde{\beta}$	$\tilde{\beta} = \beta - \theta_d$
$B^\pm \to DK^\pm$	$\tilde{\gamma}$	$\cos 2\tilde{\gamma}$	$\tilde{\gamma} = \gamma$

Table 1: The CP-violating phase information to be obtained from first-round B experiments on CP violation.

trigonometric functions of these phases which will be determined by the CP asymmetries in these three decays will be $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\sin^2 \tilde{\gamma}$ (or equivalently $\cos 2\tilde{\gamma}$), respectively.

As the notation suggests, when new physics is absent, $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$ and $\tilde{\gamma} = \gamma$. However, in the presence of new physics, the situation changes. Most likely, if new physics affects CP violation in the B system, it does so by changing the phase of the neutral B-$\bar{B}$ mixing amplitudes [13]. In this paper, we shall assume that new physics enters only in this way. In the presence of this new physics,

$$\arg A(B_q \to \overline{B_q}) = \arg A(B_q \to \overline{B_q})|_{SM} + 2\theta_q \ ; \qquad q = d, s \ . \tag{6}$$

Here, $A(B_q \to \overline{B_q})$ is, of course, the $B_q \to \overline{B_q}$ amplitude, $\arg A(B_q \to \overline{B_q})|_{SM} = 2 \arg{(V_{tq}V_{tb}^*)}$ is the phase of $B_q \to \overline{B_q}$ mixing in the Standard Model, and $2\theta_q$ is the change in this phase due to new physics. When this new physics is present, the phases probed by $B_d^0(t) \to \pi^+\pi^-$ and $B_d^0(t) \to \Psi K_S$ are changed to $\tilde{\alpha} = \alpha + \theta_d$ and $\tilde{\beta} = \beta - \theta_d$, respectively [14]. The phase $\tilde{\gamma}$ probed by $B^\pm \to DK^\pm$, which does not involve neutral B mixing, remains γ. The situation is summarized in Table 1.

To test the SM, it is reasonable to assume, at least provisionally, that no new physics is present, so that $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = (\alpha, \beta, \gamma)$. Then $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ satisfy the triangle conditions: $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are all of the same sign, and $|\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}| = \pi$. In Sec. 2 (and the Appendices), we will show that when this assumption is made, the quantities $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ to be measured by the early B CP experiments always leave a twofold discrete ambiguity in the angle-set $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. We will also show that at most one of these two candidate solutions is consistent with the SM. As we shall see, there are a variety of ways of resolving the ambiguity between the two solutions. In this paper, we will refer to this as "discrete ambiguity resolution" (DAR).

In Sec. 3, we assume that new physics is present, and affects $\tilde{\alpha}$ and $\tilde{\beta}$ as indicated in Table 1. In that case, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ still satisfy one of the triangle conditions: $|\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}| = \pi$. Thus, the presence of the new physics would not be revealed by looking for a violation of this condtion. However, when θ_d is present, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ may not all be of like sign,

CP-odd relative phase in the absence of penguins [12].

in violation of the other triangle condition, Eq. (1). If one could resolve the discrete ambiguities in $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ separately, without assuming that the phases satisfy the triangle conditions, then when $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are not of like sign one would immediately discover that new physics is present. Unfortunately, the resolution of the discrete ambiguities in $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ separately involves hadronic uncertainties, and may be quite difficult [15]. In addition, when new physics is present, the several measurements whose results must be compared to completely remove the discrete ambiguity in some angle may, in reality, not all be probing the same angle.

Suppose, then, that one simply assumes provisionally that $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ do satisfy both triangle conditions, and in particular are of like sign. Under what circumstances would new physics still be uncovered? In Sec. 3 we shall see that whether or not θ_d results in $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ not being of like sign, if one assumes that they *are* of like sign, the measured quantities (Table 1, Column 3) always lead to two candidate solutions for $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. There are then three possibilities:

1. Both of the candidate $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ solutions obtained assuming $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are of like sign are consistent with the allowed (ρ, η) region in Fig. 1. As mentioned above, in Sec. 2, we will see that, in practice, this is impossible.

2. One of the candidate $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ solutions is consistent with the allowed (ρ, η) region, but the other is not.

3. Neither of the candidate $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ solutions is consistent with the allowed (ρ, η) region. In this case, it is clear that new physics is present.

Of these three possibilities, case (2) is obviously the one which causes problems. Even if physics beyond the SM is present, due to the existence of the twofold discrete ambiguity, the measurements of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ alone will not unequivocally reveal its presence. In order to know whether or not new physics is present, it will be necessary to remove the discrete ambiguity.

When the true ($\tilde{\alpha}, \tilde{\beta}$, and $\tilde{\gamma}$) are of like sign, one of the two candidates for $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. In this situation, the discrete ambiguity resolution (DAR) techniques to be discussed in Sec. 2 will select from among the two candidates the true one. When the true ($\tilde{\alpha}, \tilde{\beta}$, and $\tilde{\gamma}$) are not of like sign, neither of the candidates for $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$, and the "DAR" techniques simply select one or the other of the incorrect candidates. Suppose, now, that in the case (2) where one of the candidates is consistent with the allowed (ρ, η) region but the other is not, DAR selects as the alleged true solution the one which is inconsistent with the allowed region. Then, regardless of the signs of the true ($\tilde{\alpha}, \tilde{\beta}$, and $\tilde{\gamma}$), one would know for certain that new physics is present, and one would not have known this without the DAR. However, it might also happen that the DAR selects the candidate solution which is consistent with the allowed (ρ, η) region. Then it could be that no new physics is present and this solution represents the true angles α, β and γ in the unitarity triangle. But it could also be that new physics *is* present, but that the CP angles and θ_d are such that it remains hidden. (Note that this can occur even for large values of θ_d.) In Sec. 3 we will provide illustrative examples of all of these situations. We conclude in Sec. 4.

2 Discrete Ambiguities and Their Resolution

The CP asymmetries in the decays $B_d^0(t) \to \pi^+\pi^-$, $B_d^0(t) \to \Psi K_S$ and $B^\pm \to DK^\pm$ permit the extraction of the functions $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\sin^2\tilde{\gamma}$ (or equivalently $\cos 2\tilde{\gamma}$), respectively[2]. (An alternative way of getting at $\tilde{\gamma}$ is through the CP asymmetry in $B_s^0(t) \to D_s^\pm K^\mp$ [16]. However, even in this case, the function measured is $\sin^2\tilde{\gamma}$. We will discuss this decay in more detail below.) Thus, from these measurements, each CP angle can be obtained up to a fourfold ambiguity: if $\tilde{\alpha}_0$, $\tilde{\beta}_0$ and $\tilde{\gamma}_0$ are the true values of these angles, the values consistent with the measurements are:

$$\begin{aligned} &\tilde{\alpha}_0 ~,~ \tilde{\alpha}_0 + \pi ~,~ \frac{\pi}{2} - \tilde{\alpha}_0 ~,~ -\frac{\pi}{2} - \tilde{\alpha}_0 ~, \\ &\tilde{\beta}_0 ~,~ \tilde{\beta}_0 + \pi ~,~ \frac{\pi}{2} - \tilde{\beta}_0 ~,~ -\frac{\pi}{2} - \tilde{\beta}_0 ~, \\ &\tilde{\gamma}_0 ~,~ \tilde{\gamma}_0 + \pi ~,~ -\tilde{\gamma}_0 ~,~ -\tilde{\gamma}_0 - \pi ~. \end{aligned} \tag{7}$$

There is thus a 64-fold discrete ambiguity in the extraction of the CP-angle set $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. However, assuming that the three angles are the interior angles of a triangle, i.e. that they satisfy Eqs. (1) and (2), this discrete ambiguity can be reduced to a twofold one. Since the main purpose of the paper is to examine the consequences of this discrete ambiguity, we defer the proof of its existence to Appendix A, and simply list the possible discrete ambiguities in Table 2.

There is one point which should be noted here. Within the SM, the magnitude of the angle β is constrained to be $16^\circ \le |\beta| \le 35^\circ$ [Eq. (3)]. However, an examination of Table 2 reveals that, regardless of $Sign(\sin 2\tilde{\beta})$, at most one of the two $\tilde{\beta}$ solutions satisfies this constraint (and it can be that neither does). Therefore, regardless of the signs of the candidate angle sets or of B_K and B_{B_d}, *at most one of the two discretely ambiguous solutions can be consistent with the SM.* This will have important consequences in our discussion of new physics.

[2]When an isospin analysis is used to extract $\sin 2\tilde{\alpha}$ from the decay $B_d^0(t) \to \pi^+\pi^-$ despite the presence of penguins, the net result is that $\sin 2\tilde{\alpha}$ is itself obtained with a fourfold discrete ambiguity, which depends on the relative magnitude and phase of the penguin and tree amplitudes. In this paper we ignore this ambiguity, since in general only one of the four values of $\sin 2\tilde{\alpha}$ yields values of $\tilde{\alpha}$ which can satisfy $|\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}| = \pi$. Furthermore, $\sin 2\tilde{\alpha}$ can be extracted independently with no discrete ambiguity from a study of $B \to \rho\pi$ decays [9].

$Sign(\sin 2\tilde{\alpha})$	$Sign(\sin 2\tilde{\beta})$	Discrete Ambiguity
>0	>0	$\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\right) \to \left(\frac{\pi}{2}-\tilde{\alpha},\frac{\pi}{2}-\tilde{\beta},\pi-\tilde{\gamma}\right)$
>0	<0	$\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\right) \to \left(-\frac{\pi}{2}-\tilde{\alpha},\frac{\pi}{2}-\tilde{\beta},-\tilde{\gamma}\right)$
<0	>0	$\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\right) \to \left(\frac{\pi}{2}-\tilde{\alpha},-\frac{\pi}{2}-\tilde{\beta},-\tilde{\gamma}\right)$
<0	<0	$\left(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\right) \to \left(-\frac{\pi}{2}-\tilde{\alpha},-\frac{\pi}{2}-\tilde{\beta},-\pi-\tilde{\gamma}\right)$

Table 2: The twofold discrete ambiguity in $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ remaining after measurement of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$.

If one assumes that there is no new physics (i.e. $\tilde{\alpha}=\alpha$, $\tilde{\beta}=\beta$, $\tilde{\gamma}=\gamma$), this twofold discrete ambiguity does not pose a problem. Since only one of the two solutions can be consistent with the SM, then clearly that is the one which must be chosen.

On the other hand, if one allows for the possibility that new physics may affect the CP asymmetries, then there may be a problem. If both solutions are inconsistent with the SM, then it is clear that new physics is present. However, if one solution is consistent with the SM, while the other is not, then the measurements of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ alone will not tell us which of the two solutions is the correct one. In other words, at this stage we will not know whether or not new physics is present. In order to decide, it will be important to be able to remove the discrete ambiguities. We examine various ways of doing this in the following subsections. For the most part, we concentrate on methods which require no knowledge of hadronic physics.

2.1 $\cos 2\tilde{\alpha}$

Another method for obtaining $\sin 2\tilde{\alpha}$ involves examining the time-dependent Dalitz plots of $B_d^0(t) \to \pi^+\pi^-\pi^0$ [9]. This final state can be fed by the intermediate states $\rho^+\pi^-$, $\rho^-\pi^+$ and $\rho^0\pi^0$. We denote the Breit-Wigner kinematic-distribution functions for the pions produced in the decays of ρ^+, ρ^- and ρ^0 as f^+, f^- and f^0, respectively (a detailed discussion of the form of the Breit-Wigner function can be found in Ref. [6]). Ignoring nonresonant contributions, the amplitude for $B_d^0 \to \pi^+\pi^-\pi^0$ can then be written

$$a \equiv Amp(B_d^0 \to \pi^+\pi^-\pi^0) = f^+a^{+-} + f^-a^{-+} + \frac{1}{2}f^0a^{00} , \tag{8}$$

while that for the CP-conjugate process is

$$\bar{a} \equiv Amp(\overline{B_d^0} \to \pi^+\pi^-\pi^0) = f^-\bar{a}^{-+} + f^+\bar{a}^{+-} + \frac{1}{2}f^0\bar{a}^{00} , \tag{9}$$

where a^{+-} and $\bar{a}^{+-}$ are, respectively, the amplitudes for a B_d^0 or $\overline{B_d^0}$ to decay into the final state $\rho^+\pi^-$. The other a^i and $\bar{a}^i$ amplitudes are defined analogously.

It is convenient to redefine the $B \to \rho\pi$ amplitudes via $A^i \equiv e^{i\tilde{\beta}} a^i$ and $\bar{A}^i \equiv e^{-i\tilde{\beta}} \bar{a}^i$. The full $B \to \pi^+\pi^-\pi^0$ amplitudes are thus similarly modified: $A \equiv e^{i\tilde{\beta}} a$ and $\bar{A} \equiv e^{-i\tilde{\beta}} \bar{a}$. The time-dependent decay rate for $B_d^0(t) \to \pi^+\pi^-\pi^0$ can then be written as

$$\Gamma(B_d^0(t) \to f) = e^{-\Gamma_B t} \left[\frac{|A|^2 + |\bar{A}|^2}{2} + \frac{|A|^2 - |\bar{A}|^2}{2} \cos(\Delta M_B t) - \mathrm{Im}\left(A^* \bar{A}\right) \sin(\Delta M_B t) \right], \tag{10}$$

where $B_d^0(t)$ is a B-meson which at $t = 0$ was a B_d^0.

Using isospin, the A^i and $\bar{A}^i$ amplitudes are related as follows [9]:

$$\frac{A^{+-} + A^{-+} + A^{00}}{\bar{A}^{+-} + \bar{A}^{-+} + \bar{A}^{00}} = e^{-2i\tilde{\alpha}} . \tag{11}$$

Thus, if one could measure the magnitudes and relative phases of the A^i and $\bar{A}^i$ amplitudes, this would provide enough information to extract the CP phase $2\tilde{\alpha}$ with no discrete ambiguity (i.e. we would know the values of both $\sin 2\tilde{\alpha}$ and $\cos 2\tilde{\alpha}$).

In fact, this is possible. The full time-dependent decay rate in Eq. (10) involves all intermediate $B \to \rho\pi$ amplitudes, which of course interfere among themselves. By itself, this rate does not give information about the individual A^i and $\bar{A}^i$ amplitudes. However, by looking at certain areas of the time-dependent $\pi^+\pi^-\pi^0$ Dalitz plot, it is possible to isolate interferences of particular pairs of $B \to \rho\pi$ amplitudes. In fact, by combining all the Dalitz-plot information, one can measure the magnitudes and relative phases of all A^i and $\bar{A}^i$ amplitudes. This holds even if there are significant penguin contributions. Thus, in addition to measuring $\sin 2\tilde{\alpha}$, this method also allows the extraction of $\cos 2\tilde{\alpha}$.

In Eq. (10), the $\mathrm{Im}\left(A^* \bar{A}\right) \sin(\Delta M_B t)$ term arises from the interference of the $B_d^0 \to \pi^+\pi^-\pi^0$ and $B_d^0 \to \overline{B_d^0} \to \pi^+\pi^-\pi^0$ decay paths, and its sign in this equation is given assuming that the bag parameter B_{B_d} is positive. Should B_{B_d} actually be negative, then the sign of $\mathrm{Im}\left(A^* \bar{A}\right)$ will be wrong. This will affect the above method since this term is used in different areas of the Dalitz plot to obtain the relative phase of the A^i and $\bar{A}^i$ amplitudes. In particular, since the minus sign (due to the wrong sign of B_{B_d}) is equivalent to an additional phase π, what is really measured in Eq. (11) is not $2\tilde{\alpha}$, but $2\tilde{\alpha} + \pi$. In other words, the measurements of *both* $\sin 2\tilde{\alpha}$ and $\cos 2\tilde{\alpha}$ will have the wrong sign.

2.2 $\cos 2\tilde{\beta}$

There are two distinct methods for measuring the function $\cos 2\tilde{\beta}$: cascade mixing and Dalitz-plot analyses. We discuss each of these in turn.

2.2.1 Cascade Mixing

In the decay chain $B_d^0 \to \Psi + K \to \Psi + (\pi^-\ell^+\nu)$, neutral K mixing follows neutral B mixing. As a result of this "cascade mixing" [17, 18], there are many paths from the initial

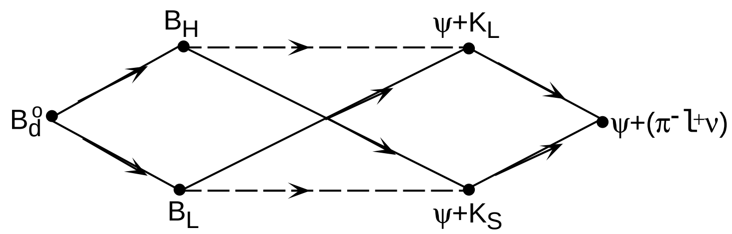

Figure 2: The paths from the initial state B_d^0 to the final state $\Psi+(\pi^-\ell^+\nu)$, where $(\pi^-\ell^+\nu)$ comes from decay of a neutral K.

state to the final one. These are shown in Fig. (2), where B_H and B_L are, respectively, the heavier and the lighter of the neutral B mass eigenstates.

Assuming that the bag parameter B_{B_d} is positive, in the limit that $\tilde{\beta} \to 0$, $\mathrm{CP}(B_H) = -\mathrm{CP}(B_L) = -1$. Consequently, in this limit the decays $B_{H(L)} \to \Psi K_{S(L)}$ are CP-conserving, while $B_{H(L)} \to \Psi K_{L(S)}$ are CP-violating and hence forbidden. Thus, it is not surprising that when $\tilde{\beta} \neq 0$, the amplitudes for $B_{H(L)} \to \Psi K_{S(L)}$ are proportional to $\cos\tilde{\beta}$, while those for $B_{H(L)} \to \Psi K_{L(S)}$ are proportional to $\sin\tilde{\beta}$. As a result, the $(B_H \to \Psi K_S) - (B_L \to \Psi K_L)$ interference probes $\cos^2\tilde{\beta}$, while the $(B_H \to \Psi K_L) - (B_L \to \Psi K_S)$ interference probes $\sin^2\tilde{\beta}$. Obviously, a suitable linear combination of these interferences then probes $\cos 2\tilde{\beta}$.

By explicit calculation one finds that [18]

$$\begin{aligned}
\Gamma & \left[B_d^0 \to \Psi + K \to \Psi + (\pi^-\ell^+\nu)\right] \\
& \propto e^{-\Gamma_B \tau_B} \Big\{ e^{-\gamma_S \tau_K} \left[1 - \sin 2\tilde{\beta}\, \sin(\Delta M_B \tau_B)\right] \\
& \qquad + e^{-\gamma_L \tau_K} \left[1 + \sin 2\tilde{\beta}\, \sin(\Delta M_B \tau_B)\right] \\
& \quad + 2\, e^{-\frac{1}{2}(\gamma_S + \gamma_L)\tau_K} \left[\cos(\Delta M_B \tau_B)\, \cos(\Delta M_K \tau_K) \right. \\
& \qquad \left. + \cos 2\tilde{\beta}\, \sin(\Delta M_B \tau_B)\, \sin(\Delta M_K \tau_K)\right] \Big\} \ . \qquad (12)
\end{aligned}$$

Here, Γ_B is the width of the B, γ_S and γ_L are, respectively, the widths of K_S and K_L, ΔM_B and ΔM_K are the positive B and K mass splittings, and τ_B and τ_K are the proper times that the B and K live before decay. A similar expression holds when the initial state is a $\overline{B_d^0}$, or the final state is $\Psi + (\pi^+\ell^-\bar{\nu})$. Note that, although the expression of the rate for $B_d^0 \to \Psi + (\pi^-\ell^+\nu)$ is more complicated than that for $B_d^0 \to \Psi K_S$, it is still independent of hadronic uncertainties. It depends only on the CP angle $\tilde{\beta}$, and on the known quantities Γ_B, γ_S, γ_L, ΔM_B and ΔM_K. The key point is that the function $\cos 2\tilde{\beta}$ appears in the expression for the rate. Thus, this method allows one to measure $\cos 2\tilde{\beta}$ and thus remove the discrete ambiguity of Table 2.

As in the probe of $\cos 2\tilde{\alpha}$ discussed in Sec. 2.1, so here it is the presence of extra interferences beyond those encountered in the simplest case that enables one to determine the cosine of a CP phase. Also, as in the probe of $\cos 2\tilde{\alpha}$, if the bag parameter B_{B_d} is actually negative, but $\mathrm{Sign}(\cos 2\tilde{\beta})$ is extracted from the data assuming $B_{B_d} > 0$, an incorrect result will be obtained. In particular, Eq. (12) assumes that $B_{B_d} > 0$.

2.2.2 $B \to D^+ D^- K_S$ and $B \to D_{CP} \pi^+ \pi^-$ Dalitz plots

Previously we saw that $\cos 2\tilde{\alpha}$ can be obtained by a measurement of the time-dependent $B_d^0(t) \to \pi^+\pi^-\pi^0$ Dalitz plot, in which the final state is fed by the intermediate states $\rho^+\pi^-$, $\rho^-\pi^+$ and $\rho^0\pi^0$. The function $\cos 2\tilde{\beta}$ can be obtained in a similar way [19].

For $\cos 2\tilde{\beta}$, the decay most analagous to $B_d^0(t) \to \pi^+\pi^-\pi^0$ is $B_d^0(t) \to D^+D^-\pi^0$, where the final state is fed by the intermediate states $D^{**+}D^-$ and D^+D^{**-}. Unfortunately, this set of decays suffers from penguin pollution, and the Dalitz plot analysis in this case does not allow one to remove it. However, there are other decays, with no penguin contributions, for which a Dalitz-plot analysis can be performed.

One example is the decay $B_d^0(t) \to D^+D^-K_S$, which is fed by the two channels $B_d^0 \to D_s^{**+}D^-$ and $\overline{B_d^0} \to D_s^{**-}D^+$. A measurement of the time-dependent Dalitz plot for this decay allows one to obtain $\cos 2\tilde{\beta}$.

Another possibility is the decay $B_d^0(t) \to D_{CP}\pi^+\pi^-$, where D_{CP} is a D^0 or $\overline{D^0}$ meson which decays to a CP eigenstate (e.g. $\pi^+\pi^-$, K^+K^-, etc.). This final state is fed by several intermediate B decays: $B_d^0 \to \pi^+D^{**-}$, $\overline{B_d^0} \to \pi^-D^{**+}$, and $B_d^0, \overline{B_d^0} \to D_{CP}\rho^0$. The function $\cos 2\tilde{\beta}$ can, in principle, be extracted from the Dalitz plot for this decay, though the analysis is somewhat more complicated than the previous example. There are also contributions from the doubly-Cabibbo-suppressed decays $B_d^0 \to \pi^-D^{**+}$ and $\overline{B_d^0} \to \pi^+D^{**-}$, which have been neglected. Their inclusion would introduce a small dependence on the CP phase $2\tilde{\beta} + \tilde{\gamma}$, which could also be extracted from the Dalitz-plot analysis. However, there is a better mode for getting at $2\tilde{\beta} + \tilde{\gamma}$, as we will discuss below.

There are thus several decay modes which are susceptible to a Dalitz-plot analysis. The function $\cos 2\tilde{\beta}$ can be obtained from such an analysis, and can be used to remove the discrete ambiguity of Table 2. If the bag parameter B_{B_d} should actually be negative, then this analysis will produce an incorrect result, as was discussed in detail for the decay $B_d^0(t) \to \pi^+\pi^-\pi^0$.

2.3 $\sin 2\tilde{\gamma}$

For the measurement of the CP angle $\tilde{\gamma}$, the decay mode most commonly discussed is $B^\pm \to DK^\pm$ [10], presumably because it is accessible at dedicated B factories. However, the CP asymmetry in $B_s^0(t) \to D_s^\pm K^\mp$ also probes $\tilde{\gamma}$ [16]. The time-dependent decay rates for these decays include terms of the form

$$\sin(\tilde{\gamma} + \Delta)\sin(\Delta M_B t) \tag{13}$$

and

$$\sin(\tilde{\gamma} - \Delta)\sin(\Delta M_B t) \ , \tag{14}$$

where Δ is a strong phase. By studying the time dependence, one can extract $\sin(\tilde{\gamma} + \Delta)$ and $\sin(\tilde{\gamma} - \Delta)$. From these two quantities one can obtain $\sin^2\tilde{\gamma}$, just as in $B^\pm \to DK^\pm$.

However, it is important to be aware that there is a distinction to be made between the CP asymmetries in $B^\pm \to DK^\pm$ and $B_s^0(t) \to D_s^\pm K^\mp$: if there is new physics in B_s mixing, the CP asymmetry in $B_s^0(t) \to D_s^\pm K^\mp$ will be affected, while that in $B^\pm \to DK^\pm$ will not. That is, even in the presence of new physics, the CP asymmetry in $B^\pm \to DK^\pm$ still measures the SM γ, while $B_s^0(t) \to D_s^\pm K^\mp$ measures $\tilde{\gamma} \neq \gamma$. Thus, if the CP asymmetries in these two decay modes are not equal, this immediately signals the presence of new physics, specifically in the B_s^0-$\overline{B_s^0}$ mixing amplitude. No discrete ambiguity resolution is necessary at all.

Hereafter we assume that, even if new physics is present, the CP asymmetry in $B_s^0(t) \to D_s^\pm K^\mp$ is unaffected, so that it probes the CP angle $\tilde{\gamma} = \gamma$, just like $B^\pm \to DK^\pm$.

The technique described above for obtaining $\sin^2\tilde{\gamma}$ from $B_s^0(t) \to D_s^\pm K^\mp$ assumes that the two B_s mass eigenstates have a common width. In fact, however, it is expected that there is a width difference between the light and heavy states of about 20% [20]. It is possible that such a large width difference is measurable at hadronic colliders. If so, additional phase information can be obtained. In the presence of a nonzero B_s width difference, the time-dependent rates also include terms of the form

$$\cos(\tilde{\gamma} + \Delta)(e^{-\Gamma_L t} - e^{-\Gamma_H t}) \tag{15}$$

and

$$\cos(\tilde{\gamma} - \Delta)(e^{-\Gamma_L t} - e^{-\Gamma_H t}) \ , \tag{16}$$

where Γ_L and Γ_H are the widths of the light and heavy mass eigenstates, respectively. These terms disappear in the limit of vanishing width difference. But if the width difference is measurable, once can also extract the functions $\cos(\tilde{\gamma} + \Delta)$ and $\cos(\tilde{\gamma} - \Delta)$.

The measurement of these four functions of $\tilde{\gamma}$ and Δ allows one to obtain $\sin 2\tilde{\gamma}$. Denoting

$$\begin{aligned} S &\equiv \sin(\tilde{\gamma} + \Delta) \quad , \quad \bar{S} \equiv \sin(\tilde{\gamma} - \Delta) \ , \\ C &\equiv \cos(\tilde{\gamma} + \Delta) \quad , \quad \bar{C} \equiv \cos(\tilde{\gamma} - \Delta) \ , \end{aligned} \tag{17}$$

we have

$$\sin 2\tilde{\gamma} = \frac{\bar{C}^2 + S^2 - C^2 - \bar{S}^2}{2(\bar{C}S - C\bar{S})} \ . \tag{18}$$

The knowledge of $\sin 2\tilde{\gamma}$ suffices to remove the discrete ambiguity in Table 2.

In fact, the measurement of $\cos 2\tilde{\gamma}$ is not even necessary. Assuming that $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ obey the triangle conditions [Eqs. (1) and (2)], except for certain singular values of the

CP angles, the measurements of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\sin 2\tilde{\gamma}$ determine $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ uniquely. This fact was also pointed out in Ref. [4], though the proof was not given. We present the proof in Appendix B.

If the bag parameter B_{B_d} should actually be negative, then, as was the case in the $B_d^0(t) \to \pi^+\pi^-\pi^0$ example, the sign of the coefficient of the $\sin(\Delta M t)$ term will be wrong. Thus, the quantities S and $\bar{S}$ will have the wrong sign [see Eqs. (13) and (14)], but C and $\bar{C}$ will be unaffected [see Eqs. (15) and (16)]. From Eq. (18), one sees that this leads to the wrong sign for $\sin 2\tilde{\gamma}$.

2.4 $\sin 2(2\tilde{\beta} + \tilde{\gamma})$

There are several CP asymmetries that probe the weak phase $2\tilde{\beta} + \tilde{\gamma}$: $B_d^0(t) \to D^{(**)} K_S$, $B_d^0(t) \to D^{\mp}\pi^{\pm}$ [21]. The function which is extracted from these asymmetries is $\sin^2(2\tilde{\beta} + \tilde{\gamma})$. It is straightforward to show that the discrete ambiguities in Table 2 are *not* resolved by this measurement.

However, there is a Dalitz-plot method which probes $2\tilde{\beta} + \tilde{\gamma}$, and which can be used to remove the discrete ambiguity [19]. The decay $B_d^0(t) \to D^{\pm}\pi^{\mp} K_S$ is fed by $B_d^0 \to \pi^- D_s^{**+}$, $\overline{B_d^0} \to D^+ K^{*-}$ and $B_d^0, \overline{B_d^0} \to D^{**0} K_S$. A Dalitz-plot analysis allows one to extract the four quantities $\sin(2\tilde{\beta} + \tilde{\gamma} + \delta)$, $\sin(2\tilde{\beta} + \tilde{\gamma} - \delta)$, $\cos(2\tilde{\beta} + \tilde{\gamma} + \delta)$ and $\cos(2\tilde{\beta} + \tilde{\gamma} - \delta)$, where δ is a strong phase. Following an analysis similar to that described in Eqs. (17) and (18), this allows one to obtain $\sin 2(2\tilde{\beta} + \tilde{\gamma})$. And knowledge of this function *is* sufficient to resolve the twofold ambiguity of Table 2.

As in the other Dalitz-plot analyses, if the bag parameter B_{B_d} should actually be negative, this method yields the wrong sign for $\sin 2(2\tilde{\beta} + \tilde{\gamma})$.

2.5 Reduction of the Allowed (ρ, η) Region

As discussed above, the main problem caused by the presence of the twofold discrete ambiguity is the possibility of having one solution inside the allowed region of Fig. 1, and the other outside. In this case, one does not know whether or not new physics is present. In the subsections above, we examined measurements which can be used to remove the discrete ambiguity. However, there is another approach which can be used. If the allowed region of Fig. 1 were reduced, this would then reduce the likelihood of having one solution inside the allowed region in the presence of new physics.

The measurements which contribute to the allowed region of Fig. 1 are $|V_{cb}|$, $|V_{ub}/V_{cb}|$, B_d and B_s mixing, and CP violation in the kaon system (ϵ_K). If the error on any of these measurements can be reduced, either through reduced experimental error or better understanding of the theoretical uncertainties, this would help to reduce the allowed region. In fact, over the past year or two, the improved lower bound on B_s mixing has already removed roughly half of the previously-allowed (ρ, η) region. An actual value for this mixing would be helpful indeed.

There is another measurement which can help to reduce the allowed (ρ, η) region. Within the SM, the decay $K_L \to \pi^0 \nu\bar{\nu}$ probes the Wolfenstein parameter η. In the Wolfenstein parametrization, the branching ratio can be written [22]:

$$B(K_L \to \pi^0 \nu\bar{\nu}) = 3.0 \times 10^{-11} \left[\frac{\eta}{0.39}\right]^2 \left[\frac{\bar{m}_t(m_t)}{170 \text{ GeV}}\right]^{2.3} \left[\frac{|V_{cb}|}{0.040}\right]^4 . \qquad (19)$$

Since the branching ratio is proportional to η^2, its measurement would greatly help in constraining the allowed (ρ, η) region.

Note that the precision with which η can be extracted from a measurement of $B(K_L \to \pi^0 \nu\bar{\nu})$ is mainly limited by the error on $|V_{cb}|$, which presently stands at about 4.3% [1], including both the experimental and theoretical (HQET) errors. If the error on $|V_{cb}|$ could be reduced, then this measurement could be used to obtain $|\eta|$, the height of the unitarity triangle.

3 New Physics

Let us now assume that CP violation in the B system is affected by new physics (NP) beyond the Standard Model. We assume, in particular, that the NP affects CP-violating asymmetries by modifying the phase of B_d^0-$\overline{B_d^0}$ mixing, as described in the Introduction [Eq. (6)]. The phases probed by the first-round CP experiments on the B system are then the quantities $\tilde{\alpha} = \alpha + \theta_d$, $\tilde{\beta} = \beta - \theta_d$ and $\tilde{\gamma} = \gamma$ given in the last column of Table 1.

As we have noted, while $|\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}| = \pi$, in the presence of a nonzero θ_d, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ may not all be of the same sign, unlike the true angles α, β and γ in the unitarity triangle. Resolving the discrete ambiguities in $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ separately, and finding that they are not all of like sign, would reveal the presence of NP. However, as already mentioned, resolving the ambiguities in $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ separately would be difficult. Therefore, we ask under what conditions the NP would still be visible even if one proceeds by simply *assuming* that $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ satisfy both of the "triangle conditions" [Eqs. (1) and (2)].

To answer this question, we first show that, given measured values of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$, there are always two candidate solutions for $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ which satisfy both triangle conditions. We call such solutions "triangle angle sets." To demonstrate that two of them always exist, we distinguish two possibilities:

1. Suppose that, even with a nonzero θ_d, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are all of the same sign (that is, positive). Then the angle set $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ clearly satisfies the two triangle conditions of Eqs. (1) and (2). There is then a second triangle angle set with the same values of the measured quantities. It is related to $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ by one of the entries in Table 2. Note that one of these two triangle angle sets is obviously the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

Angle(s) flipped	$(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$
none	$\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\right)$
α	$\left(\tilde{\alpha}, \tilde{\beta} - \pi, \tilde{\gamma} - \pi\right)$
β	$\left(\tilde{\alpha} - \pi, \tilde{\beta}, \tilde{\gamma} - \pi\right)$
α, β (1)	$\left(\tilde{\alpha} - \pi, \tilde{\beta} + \pi, \tilde{\gamma}\right)$
α, β (2)	$\left(\tilde{\alpha} + \pi, \tilde{\beta} - \pi, \tilde{\gamma}\right)$

Table 3: Construction of the triangle angle set $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ in the presence of NP. The second triangle angle set $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ can be obtained from the appropriate entry in Table 2. (1) refers to the case where $\pi < \tilde{\alpha} < 2\pi$ and $-\pi < \tilde{\beta} < 0$; (2) refers to $-\pi < \tilde{\alpha} < 0$ and $\pi < \tilde{\beta} < 2\pi$.

2. Suppose, instead, that θ_d is such that $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are not of the same sign. This can occur in a number of ways. For example, θ_d can be such that $\tilde{\alpha} = \alpha + \theta_d$ satisfies $-\pi < \tilde{\alpha} < 0$, while $\tilde{\beta} = \beta - \theta_d$ satisfies $0 < \tilde{\beta} < \pi$. Or perhaps the result is $0 < \tilde{\alpha} < \pi$ and $-\pi < \tilde{\beta} < 0$. We refer to these situations, where θ_d has flipped the sign of α or β, but not both, as a "single flip." (Of course, $\tilde{\gamma} = \gamma$ is unaffected by θ_d, and so remains positive.) It is also possible that θ_d flips the sign of both α and β – a "double flip." In all cases, one ends up with two of the three CP angles being of like sign, with the third having the opposite sign.

   In all of these cases, to obtain a triangle angle set, one has simply to add $\pm\pi$ to the two angles which have the same sign. This will give three same-sign angles which form a triangle and reproduce the measured quantities. And, as above, the second triangle angle set is obtained from the appropriate entry in Table 2. However, in contrast to the previous case, neither of these two candidate triangle angle sets is the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

In Table 3 we summarize the relation between the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ and the triangle set $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ most simply related to it.

In summary, if NP enters by modifying B_d^0-$\overline{B_d^0}$ mixing, then there are always two triangle angle sets, $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ and $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$, which give *any* observed values of the measured quantities. These two sets are related by one of the entries in Table 2, depending on the signs of $\sin 2\tilde{\alpha}$ and $\sin 2\tilde{\beta}$. If the true $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are of like sign (no flips), then one of these two angle sets, which we call $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$, is the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. If the true $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are not of like sign (one or two flips), then obviously neither $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ nor $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ is the true $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

Suppose, now, that $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ are measured. Suppose further that one then proceeds by assuming the underlying angles $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ to be the angles α, β and γ in the unitarity triangle, and looking for inconsistencies. One is then assuming that

$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is a triangle angle set, so one would calculate that $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is either the set of angles we have called $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$, or the one we have called $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$. As already stated in the Introduction, one will then encounter one of the following three situations:

1. Both $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ and $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ are consistent with the allowed (ρ, η) region in Fig. 1.

2. Only one of $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ and $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ is consistent with the allowed (ρ, η) region.

3. Neither $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ nor $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ is consistent with the allowed (ρ, η) region.

As we have already argued in Sec. 2, in practice situation (1) can never occur. Only one of the two solutions related as in Table 2 will be consistent with known physics. However, if new physics is indeed present, then resolution of the discrete ambiguity can sometimes establish its presence, as we would like to illustrate by some examples.

We are assuming that new physics affects CP violation in the B system only by changing the phase of neutral B mixing. We have seen that when this is the case, there are always two candidate triangle angle sets consistent with any given values of the measured quantities $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$. Thus, if no other quantities are measured, one cannot uncover the presence of the NP by trying to determine whether the angles underlying the measured $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ form a triangle angle set. Can one, alternatively, uncover it by determining that the candidate triangle angle sets correspond to points (ρ, η) which are outside the allowed region? By considering a large number of examples, summarized in Tables 4-8, we find that frequently one can, but not always. Tables 4, 5, 6, 7 and 8 correspond, respectively, to the assumption that the true angles in the unitarity triangle correspond to a point (ρ, η) which is near the center of the (ρ, η) region allowed by Fig. 1, near the left edge of this region, right edge, the top, and the bottom[3]. For each of these five sample cases, we explore the effect of a θ_d of 22°, 45°, 78° and 90°. We give for each θ_d the two candidate triangle angle sets, $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ and $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$, that would be inferred from the measured $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$. Next to each angle set, we give the corresponding inferred (ρ, η) point.

Each of Tables 4-8 covers values of $2\theta_d$, the angle which actually enters in $A(B_d^0 \to \overline{B_d^0})$ [see Eq. (6)], spanning the full range $(0, \pi)$. The effect of a negative $2\theta_d$ in the range $(-\pi, 0)$ can be deduced from that of the positive angle $2\theta_d + \pi$, since $2\theta_d$ leads to values of $\sin 2\tilde{\alpha}$ and $\sin 2\tilde{\beta}$ opposite to those produced by $2\theta_d + \pi$. Thus, the candidate triangle angle sets

[3]In all of our examples, we assume that the true (α, β, γ) correspond to a (ρ, η) within or very near the allowed (ρ, η) region of Fig. 1. However, it should be noted that since the new physics affects B mixing, which is one of the inputs to Fig. 1, that region may not represent the true allowed (ρ, η) region. In other words, in the presence of new physics, the true SM (α, β, γ) may already lie outside the region of Fig. 1. In this paper, we do not consider this additional possibility, but its inclusion would not affect our conclusions.

θ_d	$\tilde{\alpha}_1$, $\tilde{\beta}_1$, $\tilde{\gamma}_1$	(ρ, η)	$\tilde{\alpha}_2$, $\tilde{\beta}_2$, $\tilde{\gamma}_2$	(ρ, η)
22°	117°, 3°, 60°	$(0.03, 0.05)$	-27°, -93°, -60°	$(1.10, -1.90)$
45°	-40°, -20°, -120°	$(-0.27, -0.46)$	-50°, -70°, -60°	$(0.61, -1.06)$
68°	-17°, -43°, -120°	$(-1.17, -2.02)$	-73°, -47°, -60°	$(0.38, -0.66)$
90°	5°, 115°, 60°	$(5.20, 9.01)$	-95°, -25°, -60°	$(0.21, -0.37)$

Table 4: The effects of new physics when the true angles in the unitarity triangle are $(\alpha, \beta, \gamma) = (95^\circ, 25^\circ, 60^\circ)$, corresponding to $(\rho, \eta) = (0.21, 0.37)$.

θ_d	$\tilde{\alpha}_1$, $\tilde{\beta}_1$, $\tilde{\gamma}_1$	(ρ, η)	$\tilde{\alpha}_2$, $\tilde{\beta}_2$, $\tilde{\gamma}_2$	(ρ, η)
22°	-83°, -2°, -95°	$(0.0, -0.04)$	-7°, -88°, -85°	$(0.71, -8.17)$
45°	-60°, -25°, -95°	$(-0.04, -0.49)$	-30°, -65°, -85°	$(0.16, -1.81)$
68°	-37°, -48°, -95°	$(-0.11, -1.23)$	-53°, -42°, -85°	$(0.07, -0.83)$
90°	-15°, -70°, -95°	$(-0.32, -3.62)$	-75°, -20°, -85°	$(0.03, -0.35)$

Table 5: Same as Table 4, but with the true $(\alpha, \beta, \gamma) = (75^\circ, 20^\circ, 85^\circ)$, corresponding to $(\rho, \eta) = (0.03, 0.35)$.

θ_d	$\tilde{\alpha}_1$, $\tilde{\beta}_1$, $\tilde{\gamma}_1$	(ρ, η)	$\tilde{\alpha}_2$, $\tilde{\beta}_2$, $\tilde{\gamma}_2$	(ρ, η)
22°	132°, 8°, 40°	$(0.14, 0.12)$	-42°, -98°, -40°	$(1.13, -0.95)$
45°	-25°, -15°, -140°	$(-0.47, -0.39)$	-65°, -75°, -40°	$(0.82, -0.69)$
68°	-2°, -38°, -140°	$(-13.51, -11.34)$	-88°, -52°, -40°	$(0.60, -0.51)$
90°	20°, 120°, 40°	$(1.94, 1.63)$	-110°, -30°, -40°	$(0.41, -0.34)$

Table 6: Same as Table 4, but with the true $(\alpha, \beta, \gamma) = (110^\circ, 30^\circ, 40^\circ)$, corresponding to $(\rho, \eta) = (0.41, 0.34)$.

θ_d	$\tilde{\alpha}_1$, $\tilde{\beta}_1$, $\tilde{\gamma}_1$	(ρ, η)	$\tilde{\alpha}_2$, $\tilde{\beta}_2$, $\tilde{\gamma}_2$	(ρ, η)
22°	107°, 8°, 65°	$(0.06, 0.13)$	-17°, -98°, -65°	$(1.43, -3.07)$
45°	-50°, -15°, -115°	$(-0.14, -0.31)$	-40°, -75°, -65°	$(0.64, -1.36)$
68°	-27°, -38°, -115°	$(-0.57, -1.23)$	-63°, -52°, -65°	$(0.37, -0.80)$
90°	-5°, -60°, -115°	$(-4.20, -9.01)$	-85°, -30°, -65°	$(0.21, -0.45)$

Table 7: Same as Table 4, but with the true $(\alpha, \beta, \gamma) = (85^\circ, 30^\circ, 65^\circ)$, corresponding to $(\rho, \eta) = (0.21, 0.45)$.

θ_d	$\tilde{\alpha}_1$, $\tilde{\beta}_1$, $\tilde{\gamma}_1$	(ρ, η)	$\tilde{\alpha}_2$, $\tilde{\beta}_2$, $\tilde{\gamma}_2$	(ρ, η)
$22°$	$-45°$, $-5°$, $-130°$	$(-0.08, -0.09)$	$-45°$, $-85°$, $-50°$	$(0.91, -1.08)$
$45°$	$-22°$, $-28°$, $-130°$	$(-0.81, -0.96)$	$-68°$, $-62°$, $-50°$	$(0.61, -0.73)$
$68°$	$1°$, $129°$, $50°$	$(28.62, 34.11)$	$-91°$, $-39°$, $-50°$	$(0.40, -0.48)$
$90°$	$23°$, $107°$, $50°$	$(1.57, 1.87)$	$-113°$, $-17°$, $-50°$	$(0.20, -0.24)$

Table 8: Same as Table 4, but with the true $(\alpha, \beta, \gamma) = (113°, 17°, 50°)$, corresponding to $(\rho, \eta) = (0.20, 0.24)$.

$(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ and $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ which correspond to a NP angle $2\theta_d$ are just the negatives of those for $2\theta_d + \pi$.

In all of these Tables, the triangle angle set $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$ is found as described in Table 3. That is, $\tilde{\alpha}_1$ is always equal to $\tilde{\alpha}$, $\tilde{\alpha} + \pi$ or $\tilde{\alpha} - \pi$, and similarly for $\tilde{\beta}_1$ and $\tilde{\gamma}_1$. The second angle set $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ is obtained from the appropriate entry in Table 2. This is important when one considers the resolution of the twofold discrete ambiguity. In Secs. 2.1-2.4, we described various measurements which can be used for DAR. In all cases, one of the following trigonometric functions is obtained: $\cos 2\tilde{\alpha}$, $\cos 2\tilde{\beta}$, $\sin 2\tilde{\gamma}$ or $\sin 2(2\tilde{\beta} + \tilde{\gamma})$. The key point is that all of these functions are unchanged when one replaces $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ by $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$. On the other hand, these functions change sign if $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ is used. In other words, *in all cases, the resolution of the discrete ambiguity chooses the angle set* $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$.

Perusal of Tables 4-8 reveals several interesting points. First, we note that most of the candidate angle sets in the survey represented by these Tables correspond to (ρ, η) points well outside both the allowed (ρ, η) region in Fig. 1 and its $\eta \to -\eta$ mirror image. Whenever both candidate angle sets for a given true (α, β, γ) and θ_d have (ρ, η) values outside the allowed region, accurate measurements of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ would make the presence of new physics clear. We notice, however, that some of the candidate angle sets in Tables 4-8 have (ρ, η) values rather close to, or actually inside, the mirror image of the allowed region. This means that the candidates obtained when $2\theta_d$ is replaced by $2\theta_d - \pi$ would have (ρ, η) values inside the allowed region itself. Whenever this happens, one would not know new physics is present without measuring additional quantities.

As Tables 4-8 illustrate, if $\theta_d = \frac{\pi}{2}$, the candidate angle set $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$ is always identical to the true (α, β, γ), except that the unitarity triangle has been flipped over (the common sign of the angles has been reversed, and η has been replaced by $-\eta$). The reason for this is simply that when $\theta_d = \pm\frac{\pi}{2}$, $\sin 2\tilde{\alpha} = -\sin 2\alpha$ and $\sin 2\tilde{\beta} = -\sin 2\beta$. Thus, since $\cos 2\tilde{\gamma}$ is insensitive to the sign of $\tilde{\gamma}$, one of the candidate triangles looks like the real unitarity triangle, but flipped. If one assumes that the theoretical signs of B_{B_d} and B_K are correct, the flipped character of this triangle would imply that it cannot be the true unitarity triangle. Thus, if, as in all of the $\theta_d = \frac{\pi}{2}$ examples in Tables 4-8, the other candidate triangle, $(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1)$, is also inconsistent with the allowed (ρ, η) region, one would conclude that new physics is present. This conclusion would be confirmed by resolving the discrete ambiguity, since as we have seen this resolution would always reject the triangle $(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2)$, which here is the candidate mirroring the true unitarity triangle.

As Tables 4-8 also show, values of θ_d other than $\frac{\pi}{2}$ can also lead to candidate triangles which are at least close to being consistent with the allowed (ρ, η) region of Fig. 1 or its $\eta \to -\eta$ mirror image. One interesting example of this phenomenon is the second row of Table 5, for the case $(\alpha, \beta, \gamma) = (75^\circ, 20^\circ, 85^\circ)$ and $\theta_d = 45^\circ$. From this entry, it follows that $\theta_d = -45^\circ$ would lead to the candidate triangles

$$(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1) = (30^\circ, 65^\circ, 85^\circ) \tag{20}$$

with

$$(\rho, \eta) = (0.16, 1.81) \ , \tag{21}$$

and

$$(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2) = (60^\circ, 25^\circ, 95^\circ) \tag{22}$$

with

$$(\rho, \eta) = (-0.04, 0.49) \ . \tag{23}$$

This first of these is clearly inconsistent with the allowed (ρ, η) region of Fig. 1, but the second is rather close to being consistent with it. Absent any additional information, one would not know whether new physics is present or not. However, as always, DAR would rule out candidate 2 – the triangle which is close to consistency with the Standard Model – leaving only candidate 1, which is completely inconsistent with the SM. Once again, new physics would thereby be clearly established.

In fact, by changing the parameters of this example slightly, we can easily produce a case where, despite a large θ_d, one of the candidate triangles has a (ρ, η) actually inside the allowed region. Suppose that $(\alpha, \beta, \gamma) = (70^\circ, 20^\circ, 90^\circ)$, so that $(\rho, \eta) = (0.0, 0.36)$ is inside the allowed region, and that $\theta_d = -50^\circ$. The measured $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\cos 2\tilde{\gamma}$ would then lead to the candidate triangles

$$(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1) = (20^\circ, 70^\circ, 90^\circ) \tag{24}$$

with

$$(\rho, \eta) = (0.00, 2.75) \ , \tag{25}$$

and

$$(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2) = (70^\circ, 20^\circ, 90^\circ) \tag{26}$$

with

$$(\rho, \eta) = (0.00, 0.36) \ . \tag{27}$$

Triangle 1 is completely inconsistent with the allowed (ρ, η) region, but triangle 2 happens to coincide with the true SM unitarity triangle (despite the presence of a large θ_d!), and so is totally consistent with the SM. As above, a DAR would select candidate 1, which is inconsistent with the allowed (ρ, η) region. Thus, once again, such a measurement would establish that new physics is present.

As these examples illustrate, if one of the two candidate triangles is fully consistent with the allowed (ρ, η) region and the other is far from consistent with it, the assumption that the consistent candidate represents the true CP-violating phases in B decay and no new physics is present can be completely erroneous. In these examples the resolution of the discrete ambiguity would reveal that the Standard Model-consistent candidate does *not* represent the true CP phases, and that new physics *is* present.

In these two examples, the DAR chose the candidate solution which is inconsistent with the allowed (ρ, η) region, thereby clearly revealing the presence of new physics. But it is also possible for the DAR to choose the consistent solution. As an example, consider the third line of Table 8. From this entry, it follows that if θ_d were $-22°$, the two candidate triangles would be

$$(\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1) = (91°, 39°, 50°) \tag{28}$$

with

$$(\rho, \eta) = (0.40, 0.48) \ , \tag{29}$$

and

$$(\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2) = (-1°, -129°, -50°) \tag{30}$$

with

$$(\rho, \eta) = (28.62, -34.11) \ . \tag{31}$$

Here, triangle 1 is (nearly) consistent with the allowed (ρ, η) region, while triangle 2 is not. In this situation, DAR, by ruling out triangle 2 as always, will select the consistent solution. This might tempt one to erroneously conclude that there is no new physics present. However, as this example illustrates, large new-physics effects may actually be present in CP asymmetries even when these asymmetries appear to be consistent with the SM. This is something of which one should be wary.

It is interesting to notice that the conclusion that new physics is present can sometimes depend crucially on the theoretical signs of the bag parameters B_{B_d} and B_K being correct. To see this, suppose that one of the two candidate triangles implied by the measured values of $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$, and $\cos 2\tilde{\gamma}$ is consistent with the allowed (ρ, η) region or with its $\eta \to -\eta$ mirror image, while the other candidate triangle is not. Suppose further that a DAR selects the triangle which is not consistent with either the allowed (ρ, η) region or its mirror image. If we assume that the theoretical signs of B_{B_d} and B_K are correct, as we have been doing, then we can conclude that new physics is present. Imagine, however, that we allow for the possibility that the theoretical Sign(B_{B_d}) and/or Sign(B_K) is wrong. Can we still conclude that new physics is present?

The answer to this question is "no". The main point is that our assumed techniques for determining $\sin 2\tilde{\alpha}$ and $\sin 2\tilde{\beta}$, and for DAR, all depend on an interference between a B_d^0 decay-path which involves $B_d^0 - \overline{B_d^0}$ mixing and one which does not. If the Sign(B_{B_d}) used when interpreting the data is wrong, then the assumed relation between the measured interference and the underlying parameters will be wrong. One can show that, as a result, the extracted $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$, and any of the quantities (such as $\cos 2\tilde{\alpha}$) to be used by DAR, will have the wrong sign. This means that the two candidate triangles deduced from the data will be upside down relative to what they should be, and that the DAR will select the wrong candidate triangle. Thus, when DAR appears to select the candidate triangle which is not consistent with the allowed (ρ, η) region, this could be due to Sign(B_{B_d}) being wrong, rather than to this triangle representing the truth and new physics being present. The ability to conclude unambiguously that new physics is present has been lost.[4]

The situation is summarized in detail in Table 9. In constructing this table, we have assumed that one of the two candidate triangles is consistent with the allowed (ρ, η) region or with its mirror image, that the other candidate triangle is not consistent with either, and that DAR selects the latter triangle. For all possible orientations of the candidate triangles, we indicate whether these circumstances still imply the presence of new physics when the theoretical sign of B_K and/or B_{B_d} is wrong. Table 9 makes clear that, regardless of the orientations of the candidate triangles, if the theoretical Sign(B_{B_d}) might be wrong, then one cannot unambiguously conclude that new physics is present. To be sure, if the allowed-(ρ, η)-region-consistent candidate triangle points up, then this conclusion *is* still possible if one knows for certain that the theoretical Sign(B_K) is correct. However, realistically, if one is unsure of Sign(B_{B_d}), then one is unsure of Sign(B_K) as well.

None of this is meant to cast doubt on the theoretically-determined signs of B_K and B_{B_d}. However, since these signs are not experimentally verified, it is important to recognize the crucial role that they may prove to play in establishing the existence of new physics.

From the expressions which determine the candidate triangles 1 and 2 that correspond to given (α, β, γ) and θ_d, it is relatively straightforward to show that neither candidate will be consistent with the presently-allowed (ρ, η) region unless

$$(i) \quad 80^\circ \lesssim \gamma \lesssim 100^\circ \tag{32}$$

[4] This is also true when one measures $\sin 2\tilde{\gamma}$ in $B_s^0(t) \to D_s^{\pm} K^{\mp}$, which involves $B_s^0 - \overline{B_s^0}$, rather than $B_d^0 - \overline{B_d^0}$, mixing. As we have noted, when $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$, and $\sin 2\tilde{\gamma}$ are known, there is only one candidate triangle. However, if the signs of the bag parameters used to determine this triangle are wrong, it may not represent the true $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$. Suppose, for instance, that $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = (70^\circ, 30^\circ, 80^\circ)$. If Sign($B_{B_d}$) is right, but Sign($B_{B_s}$) is wrong, we will get $(20^\circ, 60^\circ, 100^\circ)$.

Allowed (ρ, η) Consistent Candidate	Allowed (ρ, η) Inconsistent Candidate	B_K	B_{B_d}	NP Definitely Present
Up	Up	Right	Right	Yes
		Wrong	Right	Yes
		Right	Wrong	Yes
		Wrong	Wrong	No
Down	Down	Right	Right	Yes
		Wrong	Right	Yes
		Right	Wrong	No
		Wrong	Wrong	Yes
Up	Down	Right	Right	Yes
		Wrong	Right	Yes
		Right	Wrong	Yes
		Wrong	Wrong	No
Down	Up	Right	Right	Yes
		Wrong	Right	Yes
		Right	Wrong	No
		Wrong	Wrong	Yes

Table 9: The effect of incorrect signs of B_K and B_{B_d} on one's ability to conclude that new physics (NP) is present. In the first column, we indicate whether the candidate triangle consistent with the allowed (ρ, η) region or its mirror image points up ($\eta > 0$) or down ($\eta < 0$). In the second column, we show the same thing for the candidate triangle which is not consistent with either the allowed (ρ, η) region or its mirror image. In the third and fourth columns, we indicate whether the theoretical signs of B_K and B_{B_d} are right or wrong. In the final column, we indicate whether, under the stated assumptions concerning the signs of B_K and B_{B_d}, one would still be able to conclude that NP is definitely present.

or

$$(ii)\ \ 0 \leq |\theta_d| \lesssim 20^\circ \quad or \quad 70^\circ \lesssim |\theta_d| \lesssim 90^\circ \ . \tag{33}$$

However, when one of these conditions is met, or approximately met, then as the examples we have considered show, it is indeed possible for a candidate triangle to be nearly or fully consistent with the allowed region. When this occurs, one cannot establish that new physics is present without an additional measurement. Resolution of the discrete ambiguity, as described in Secs. 2.1-2.4, would often provide the needed information. However, it is important to remember that, even if this resolution chooses the SM-consistent solution, that does not completely rule out the possibility that large new-physics effects are present.

4 Conclusions

Within the standard model, CP violation is due to complex phases in the CKM matrix. This explanation will be tested through the measurement of CP-violating asymmetries in the B system. Such measurements will permit the extraction of the interior angles α, β and γ of the unitarity triangle. If the unitarity triangle constructed from these CP angles is inconsistent with the (ρ, η) region allowed by other measurements ($|V_{cb}|$, $|V_{ub}/V_{cb}|$, B_d and B_s mixing, ϵ_K), this will signal the presence of new physics.

Unfortunately, it is not the CP angles themselves which will be measured, but rather trigonometric functions of these angles. In particular, it is very likely that the first measurements will extract the functions $\sin 2\alpha$, $\sin 2\beta$ and $\sin^2\gamma$ (or equivalently $\cos 2\gamma$). This implies that the CP angles can be obtained only up to discrete ambiguities. In this paper we have demonstrated that, under the assumption that the angles are the interior angles of a triangle (i.e. that they are all of the same sign and add up to $\pm\pi$), a twofold discrete ambiguity remains in the triangle angle set (α, β, γ). That is, there are two sets of solutions which form a triangle and still reproduce the experimental results.

If one does not allow for the possibility of new physics, this discrete ambiguity causes no problems. As shown in the paper, at most one of these two solutions is consistent with present experimental constraints on the unitarity triangle. Thus, in the absence of new physics, one simply chooses that solution which is consistent with the allowed (ρ, η) region.

On the other hand, if one allows for the possibility of new physics, then there may be a problem. In the presence of new physics which modifies the phase of $B_d^0-\overline{B_d^0}$ mixing, the CP angles measured are not the SM angles α, β and γ, but rather $\tilde{\alpha} = \alpha + \theta_d$, $\tilde{\beta} = \beta - \theta_d$ and $\tilde{\gamma} = \gamma$, where θ_d is the modification due to new physics. Even in this case, though, there are still two candidate triangle angle sets. If both solutions are inconsistent with the allowed (ρ, η) region, then new physics is clearly present. But if one solution is consistent with this region, while the other is not, then one cannot be certain whether new physics is or is not present. In this case it is necessary to resolve the discrete ambiguity.

We have reviewed the various methods for discrete ambiguity resolution (DAR). They involve the measurement of different trigonometric functions of the CP angles: $\cos 2\tilde{\alpha}$, $\cos 2\tilde{\beta}$, $\sin 2\tilde{\gamma}$, or $\sin 2(2\tilde{\beta}+\tilde{\gamma})$. Any one of these measurements is sufficent to remove the twofold discrete ambiguity. (In fact, if one measures $\sin 2\tilde{\alpha}$, $\sin 2\tilde{\beta}$ and $\sin 2\tilde{\gamma}$, there is no discrete ambiguity at all in the triangle angle set – the measurement of $\cos 2\tilde{\gamma}$ is not even necessary.)

In this paper, we have presented several examples in which such a discrete ambiguity resolution is necessary. We have found a number of different values of the SM (α,β,γ) and new-physics θ_d which yield the situation in which one triangle angle set is consistent with the allowed (ρ,η) region, while the other is not. In some of these examples, the DAR chooses the solution which is inconsistent with the allowed (ρ,η) region, thereby demonstrating that new physics is present. Without DAR, one might have been led (erroneously) to think that the solution which is consistent with the allowed region is the true SM solution, and no new physics is present. However, we have also found an example in which large new-physics effects are present, but the DAR still chooses the solution which is consistent with the allowed (ρ,η) region. In such cases, the new physics remains hidden and other measurements are required to uncover it.

Note that our analysis and conclusions are strongly dependent on the size of the allowed (ρ,η) region. Any reduction in the allowed region, such as an actual measurement of B_s mixing or the measurement of $B(K_L \to \pi^0\nu\bar{\nu})$, would reduce the likelihood that one of the two candidate triangles will be consistent with the region even when new physics is present. (Indeed, the increasingly-stringent lower limits on B_s mixing have already helped in this regard.) If the region were sufficiently reduced, then, except for some very fine-tuned choices of θ_d, DAR would not be necessary at all.

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Appendix A

In this Appendix, we present the proof that, assuming that the three angles α, β and γ are the interior angles of a triangle, the functions $\sin 2\alpha$, $\sin 2\beta$ and $\cos 2\gamma$ determine the angle set (α,β,γ) up to a twofold ambiguity.

First, suppose that $\sin 2\alpha$ and $\sin 2\beta$ have the same sign, e.g. assume that they are both positive. Since α and β must have the same sign [see Eq. (1)], this implies that 2α and 2β both take values in the domain $(-2\pi,-\pi)$ or $(0,\pi)$. We can immediately exclude the $(-2\pi,-\pi)$ domain: since $|2\alpha|>\pi$ and $|2\beta|>\pi$, this implies that $|\alpha|+|\beta|>\pi$, in violation of Eq. (2). Thus, for $\phi = \alpha, \beta$ we can write

$$2\phi = \frac{\pi}{2} + 2\delta_\phi \quad , \qquad |\delta_\phi| < \frac{\pi}{4} \ . \tag{34}$$

The magnitude of δ_ϕ, *but not its sign*, is fixed by the measured value of $\sin 2\phi$. From Eq. (34) and the assumption that $|\alpha + \beta + \gamma| = \pi$, we have

$$\begin{aligned} \alpha &= \frac{\pi}{4} + \delta_\alpha \ , \\ \beta &= \frac{\pi}{4} + \delta_\beta \ , \\ \gamma &= \frac{\pi}{2} - (\delta_\alpha + \delta_\beta) \ . \end{aligned} \tag{35}$$

Now, the measured value of $\cos 2\gamma = -\cos 2(\delta_\alpha + \delta_\beta)$ gives us the relative sign of δ_α and δ_β. However, there still remains a twofold sign ambiguity, corresponding to $\delta_{\alpha,\beta} \to -\delta_{\alpha,\beta}$. This is equivalent to the twofold discrete ambiguity

$$(\alpha, \beta, \gamma) \to \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \beta, \pi - \gamma \right) . \tag{36}$$

Note that both sets of CP angles in the above discrete ambiguity correspond to unitarity triangles which point up.

A similar analysis holds when $\sin 2\alpha$ and $\sin 2\beta$ are both negative, except that in this case the discrete ambiguity is between two downward-pointing unitarity triangles:

$$(\alpha, \beta, \gamma) \to \left(-\frac{\pi}{2} - \alpha, -\frac{\pi}{2} - \beta, -\pi - \gamma \right) . \tag{37}$$

Now suppose that $\sin 2\alpha$ is positive and $\sin 2\beta$ is negative. Thus, 2α lies in the domain $(-2\pi, -\pi)$ or $(0, \pi)$, while 2β is in $(-\pi, 0)$ or $(\pi, 2\pi)$. There are now two cases to consider:

1. If α, β, γ are all positive, then

$$2\alpha = \frac{\pi}{2} + 2\delta_\alpha \ , \tag{38}$$

$$2\beta = \frac{3\pi}{2} + 2\delta_\beta \ , \tag{39}$$

where $|\delta_\alpha|, |\delta_\beta| < \pi/4$. The measured values of $\sin 2\alpha$ and $\sin 2\beta$ determine the magnitudes of δ_α and δ_β, but not their signs. From Eq. (39) and the assumption that $|\alpha + \beta + \gamma| = \pi$, we have

$$\begin{aligned} \alpha &= \frac{\pi}{4} + \delta_\alpha \ , \\ \beta &= \frac{3\pi}{4} + \delta_\beta \ , \\ \gamma &= -(\delta_\alpha + \delta_\beta) \ . \end{aligned} \tag{40}$$

The requirement that $\gamma > 0$ implies that $(\delta_\alpha + \delta_\beta) < 0$. Now, the measured value of $\cos 2\gamma = \cos 2(\delta_\alpha + \delta_\beta)$ gives us the relative sign of δ_α and δ_β. This, along with the constraint that $(\delta_\alpha + \delta_\beta) < 0$, fixes δ_α and δ_β uniquely.

2. If α, β, γ are all negative, then

$$\begin{aligned} 2\alpha &= -\frac{3\pi}{2} - 2\delta_\alpha \ , \\ 2\beta &= -\frac{\pi}{2} - 2\delta_\beta \ , \end{aligned} \tag{41}$$

with $|\delta_\alpha|, |\delta_\beta| < \pi/4$. Again, the measured values of $\sin 2\alpha$ and $\sin 2\beta$ determine the magnitudes of $\delta_{\alpha,\beta}$, but not their signs. We have

$$\begin{aligned} \alpha &= -\frac{3\pi}{4} - \delta_\alpha \ , \\ \beta &= -\frac{\pi}{4} - \delta_\beta \ , \\ \gamma &= \delta_\alpha + \delta_\beta \ . \end{aligned} \tag{42}$$

This time, since $\gamma < 0$, one again requires that $(\delta_\alpha + \delta_\beta) < 0$. As before, the measured value of $\cos 2\gamma = \cos 2(\delta_\alpha + \delta_\beta)$ gives us the relative sign of δ_α and δ_β. And again, this fixes δ_α and δ_β uniquely once one takes into account the constraint that $(\delta_\alpha + \delta_\beta) < 0$.

Thus, for the case of $\sin 2\alpha > 0$ and $\sin 2\beta < 0$, we have two possible solutions for (α, β, γ): one with positive values, and the other with negative values. Now, from the definitions of δ_α and δ_β, it is clear that the magnitudes of these quantities are the same in both cases and are determined by the measured values of $\sin 2\alpha$ and $\sin 2\beta$. Furthermore, for both solutions we have $(\delta_\alpha + \delta_\beta) < 0$, with the relative sign being determined by the measurement of $\cos 2\gamma$. Thus, the two solutions have the *same* values of δ_α and δ_β. This allows us to determine the discrete ambiguity in this case. Denoting (α, β, γ) as the positive-angle solution above, the discrete ambiguity is

$$(\alpha, \beta, \gamma) \to \left(-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \beta, -\gamma \right) . \tag{43}$$

In this case, the first solution corresponds to a unitarity triangle pointing up, while the second corresponds to one pointing down.

Finally, the analysis in the case in which $\sin 2\alpha < 0$ and $\sin 2\beta > 0$ is clear from the above: the roles of α and β are reversed, and we have the following discrete ambiguity:

$$(\alpha, \beta, \gamma) \to \left(\frac{\pi}{2} - \alpha, -\frac{\pi}{2} - \beta, -\gamma \right) . \tag{44}$$

Appendix B

In this Appendix, we present the proof that, assuming that the three angles α, β and γ are the interior angles of a triangle, the functions $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ determine the angle set (α, β, γ) uniquely.

If $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ are all measured, then either (i) all three quantities are of the same sign, or (ii) two of the three quantities are of the same sign. We consider these two possibilities in turn.

Suppose first that $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ are all positive. As in Appendix A, this implies that $0 < \alpha, \beta, \gamma < \pi/2$. We can thus write

$$\begin{aligned} \alpha &= \frac{\pi}{4} + \delta_\alpha \ , \\ \beta &= \frac{\pi}{4} + \delta_\beta \ , \\ \gamma &= \frac{\pi}{2} - (\delta_\alpha + \delta_\beta) \ , \end{aligned} \tag{45}$$

where $\Delta_i \equiv |\delta_i| < \pi/4$, $i = \alpha, \beta$. Since $\gamma < \pi/2$, the bigger of δ_α and δ_β must be positive. Suppose that it is δ_α. Then, depending on the sign of δ_β, the quantity $\delta_\alpha + \delta_\beta$ is equal to either $\Delta_\alpha + \Delta_\beta$ or $\Delta_\alpha - \Delta_\beta$. However, in general the measured value of $\sin 2\gamma = \sin 2(\delta_\alpha + \delta_\beta)$ will distinguish between these two possibilities, and so measurements of $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ will determine α, β and γ uniquely. The one exception is the singular point $\Delta_\alpha = \pi/4$, or $\alpha = \pi/2$ ($\sin 2\alpha = 0$). In this case a twofold discrete ambiguity remains. A similar analysis holds if $\Delta_\beta > \Delta_\alpha$, and also when $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ are all negative.

Now suppose that two of these quantities are positive, and the third negative, say $\sin 2\alpha,\ \sin 2\beta > 0$ and $\sin 2\gamma < 0$. This implies that $0 < \alpha, \beta < \pi/2$ and $\pi/2 < \gamma < \pi$. Thus, we have again

$$\begin{aligned} \alpha &= \frac{\pi}{4} + \delta_\alpha \ , \\ \beta &= \frac{\pi}{4} + \delta_\beta \ , \\ \gamma &= \frac{\pi}{2} - (\delta_\alpha + \delta_\beta) \ , \end{aligned} \tag{46}$$

but this time the bigger of δ_α and δ_β must be negative. Again, suppose that it is δ_α, so that $\delta_\alpha + \delta_\beta$ is equal to either $-(\Delta_\alpha + \Delta_\beta)$ or $-(\Delta_\alpha - \Delta_\beta)$. In general the measured value of $\sin 2\gamma = \sin 2(\delta_\alpha + \delta_\beta)$ will distinguish between these two possibilities, so that measurements of $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ will determine α, β and γ uniquely. This will fail only in the special case in which $\Delta_\alpha = \pi/4$, or $\alpha = \pi/2$ ($\sin 2\alpha = 0$). The case $\Delta_\beta > \Delta_\alpha$ can be treated in the same way. A similar analysis holds when any two of $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ are of one sign, with the third of opposite sign.

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